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GDTM-Padé Technique for Solving the Nonlinear Delay Differential Equations

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Abstract

The delay differential equations, which have some arguments with time-lags, are widely used in many areas of physics, engineering, economics and biology. In this work, we have applied the generalized differential transform method which is a semi numerical-analytical method, for solving a delay differential equation and used the Padé-approximation for improving the accuracy and convergence of differential transform method's solutions. Theoretical considerations are discussed. Some examples are used to illustrate the validity and the great potential of this method in solving delay differential equations. Comparisons are made between the results of the approximate solutions and exact solutions. The results show that proposed method is an effective method in solving delay differential equations.

1. Introduction

Delay differential equations (DDEs) have a wide range of application in science and engineering. They arise when the rate of change of a time-dependent process in its mathematical modeling is not only determined by its present state but also by a certain past state. It has been recognized that phenomena may have a delayed effect in a differential equation, leading to what is called a DDE. DDEs play an important role in modeling many phenomena in applied sciences including the fields as diverse as engineering, biology and economy. For example, Baker et al. [1] stated some of the application areas which include population dynamics, infectious disease, physiological and pharmaceutical kinetics, chemical kinetics, the navigational control of ships and aircraft and more general control problems. More examples of the researches dealing with DDEs can be referred to as Driver [2], Kuang [3] and Macdonald [4]. The concept of the differential transform method (DTM) was first proposed by Zhou [5] in 1986, who solved linear and nonlinear problems in electrical circuit problems. DTM obtains an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method

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is computationally time consuming for larger orders. Chen and Ho [6] developed this method for partial differential equations and Ayaz [7] applied it to the system of differential equations. During recent years, many authors have been used this method for solving various types of equations. For example, this method has been used for partial differential equations [8-10], fractional differential equations [11], difference equations [12] and Volterra integral equations [13]. Also, this method has been successfully applied for solving many types of nonlinear problems [14-17]. An important fact is that polynomials are used to approximate truncated power series. Furthermore, the singularities of polynomials cannot be seen obviously in a finite plane. Since the radius of convergence of the power series may not be large enough to contain the two boundaries, it is not always useful to use the power series [18]. Padé-approximation is applied in order to manipulate the obtained series for numerical approximations to overcome this difficulty. This technique is the best approximation for a polynomial approximation of a function into rational functions of polynomials of given order [19, 20].

In the present work, a numerical method based on the generalized differential transform method (GDTM) and Padé-approximation is proposed, and then applied to the nonlinear delay differential equations (NDDEs) which can be written as the following basic form [21]

$$u^{(m)}(t) = \sum_{j=0}^{J} \sum_{n=0}^{m-1} \mu_{jn}(t) u^{(n)}(\alpha_{jn} t + \beta_{jn}) + g(u) + f(t), \tag{1}$$

with initial condition

$$\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i, \qquad i = 0,1,\ldots,m-1, \tag{2}$$

where

$$g(u) = bu + \sum_{i=1}^{l} d_i u^{\gamma_i}, d_i \in \mathbb{R}, \gamma_i \in \mathbb{N}, \gamma_i > 1, l \in \mathbb{N}, 0 \le \alpha_{jn} \le 1, \beta_{jn} \in \mathbb{R}$$

and $^{(m)}$ in $u^{(m)}$ is considered as the m-th derivative of the function u.

The article is summarized as follows: section 2 introduces the basic theorems for GDTM and its application to Eq. (1). Section 3 describes the Padé-approximation technique briefly. In section 4, the proposed method is applied to several types of Eq. (1), and a comparison is made with the available analytic or exact solutions which have been reported in other published works in the literature. Finally, a brief conclusion is given in the last section.

2. One-Dimensional GDTM

The one-dimensional differential transform of the k-th derivative of a function $y(t)$ at the point $t = t_0$ is defined as follows

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=t_0} \tag{3}$$

where $y(t)$ is the original function and $Y(k)$ is the transformed function. The differential inverse transform of $Y(k)$ is defined as

$$y(t) = \sum_{k=0}^{\infty} Y(k)(t - t_0)^k \tag{4}$$

From Eqs. (3) and (4), we arrive at

$$y(t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} \frac{d^k y(t)}{dt^k}\bigg|_{t=t_0} \tag{5}$$

which implies that the concept of differential transform is derived from the Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which is described by the transformed equations of the original function. In real applications, the function $y(t)$ is expressed by a finite series and Eq. (4) can be written as

$$y(t) = \sum_{k=0}^{N} Y(k)(t - t_0)^k \tag{6}$$

Here, N is a sufficiently large integer. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1.

2.1. Theorems for DDEs [22, 23]

In this section, the theoretical considerations for DDEs are introduced. The following theorems give arise to the generalized differential transform of given function at $t_0 = 0$.

Theorem 1. If $y(t) = u\left(\frac{t}{a}\right), a \geq 1$, then, $Y(k) = \frac{1}{a^k} U(k)\cdot$

Table 1. The fundamental operations of DTM

Original function	Transformed function
$y(t)=u(t)\pm v(t)$	$Y(k)=U(k)\pm V(k)$
$y(t)=cu(t), c\in\mathbb{R}$	$Y(k)=cU(k)$
$y(t)=u(t)v(t)$	$Y(k)=\sum_{s=0}^{k}U(s)V(k-s)$
$y(t)=\frac{d^n u(t)}{dt^n}, n\in\mathbb{N}$	$Y(k)=\frac{(k+n)!}{k!}U(k+n)$
$y(t)=t^n$	$Y(k)=\delta(k-n)\ ,\ \delta(k-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases}$
$y(t)=e^{\lambda t}, \lambda\in\mathbb{R}$	$Y(k)=\frac{\lambda^k}{k!}$
$y(t)=\sin(\omega t+\alpha)$	$Y(k)=\frac{\omega^k}{k!}\sin(\frac{k\pi}{2}+\alpha)$
$y(t)=\cos(\omega t+\alpha)$	$Y(k)=\frac{\omega^k}{k!}\cos(\frac{k\pi}{2}+\alpha)$
$y(t)=u(t+a)$	$Y(k)=\sum_{s=k}^{N}\binom{s}{k}a^{s-k}U(s) \quad for \quad N\to\infty$
$y(t)=u_1(t)u_2(t)\ldots u_{n-1}(t)u_n(t)$	$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\ldots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_1(k_1)\times$ $U_2(k_2-k_1)\ldots U_{n-1}(k_{n-1}-k_{n-2})U_n(k-k_{n-1})$
$y(t)=u(\frac{t}{a}), a\geq 1$	$Y(k)=\sum_{k=r}^{N}(-1)^{k-r}\frac{(a-1)^{k-r}}{a^k}t_0^{k-r}\binom{k}{r}a^{k-r}U(k)$ $for\ N\to\infty$

Theorem 2. If $y(t)=u(\alpha t+\beta)$ where $\alpha,\beta\in\mathbb{R}$, then, $Y(k)=\sum_{s=k}^{N}\binom{s}{k}\alpha^k\beta^{s-k}U(s)$ for $N\to\infty$.

Theorem 3. If $y(t)=\frac{d^n u(\alpha t+\beta)}{dt^n}, n\in\mathbb{N}, \alpha,\beta\in\mathbb{R}$, then, $Y(k)=\sum_{s=k}^{N}\binom{s}{k}\alpha^k\beta^{s-k}\frac{(s+n)!}{s!}U(s+n)$ for $N\to\infty$.

Now, according to the Table 1 and above theorems, the transformed problem of Eq. (1) at $t_0=0$ can be expressed as follows

$$\frac{(k+m)!}{k!}U(k+m)=\sum_{j=0}^{J}\sum_{n=0}^{m-1}\sum_{s=0}^{k}\left(M_{jn}(s)Y_{jn}(k-s)\right)+G(k)+F(k) \tag{7}$$

where $M_{jn}(k), Y_{jn}(k), G(k)$ and $F(k)$ are transformed functions of $\mu_{jn}(t)$, $u^{(n)}(\alpha_{jn}t+\beta_{jn})$, $g(u)$ and $f(t)$, respectively. Also, the transformed initial condition of Eq. (2) is given by

$$\sum_{n=0}^{m-1} c_{in} U(n) = \lambda_i \,,\; i = 0,1,\ldots,m-1 \tag{8}$$

Based on the initial conditions in Eqs. (7) and (8), we can derive the coefficients $U(k)$ and according to Eq. (6), we obtain the GDTM approximate solution as

$$u(t) = \sum_{k=0}^{N} U(k)\, t^k \tag{9}$$

3. Padé-approximation

Padé-approximations [18-20] are widely used in computer calculations due to the fact that this technique often gives better approximation of the function than truncating its power series and it may even work where the power series does not converge. For simplicity, we denote the $[L,M]$ Padé-approximation to $f(t) = \sum_{k=0}^{\infty} a_k t^k$ by

$$f[L,M] = \frac{P_L(t)}{Q_M(t)} \tag{10}$$

where

$$\begin{aligned} P_L(t) &= p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_L t^L \\ Q_M(t) &= q_0 + q_1 t + q_2 t^2 + q_3 t^3 + \ldots + q_M t^M \end{aligned} \tag{11}$$

with the normalization condition $Q_M(0) = 1$. The coefficients of $P_L(t)$ and $Q_M(t)$ can be uniquely determined by comparing the first $(L + M + 1)$ terms of the functions $f[L,M]$ and $f(t)$. In practice, the construction of the $[L,M]$ Padé-approximation involves only algebra equations, which are solved by means of the Mathematica or Maple package. To improve the accuracy and convergence of the DTM solution of Eq. (9), the Padé-approximation is used.

4. Numerical Examples

In order to illustrate the advantages and the accuracy of the DTM-Padé technique for solving the nonlinear DDEs, we apply this method for solving three linear and nonlinear DDEs from class of Eq. (1).

Example 1: In this example, we consider Eq. (1) with $m = 4, J = 2$ and the following nonzero coefficients μ_{jn}, α_{jn} and β_{jn}

$$\mu_{00} = -2\,t\,, \mu_{10} = -t^2\,,\ \mu_{20} = 2\sin(t),\ \mu_{23} = 1,$$
$$\alpha_{00} = 1,\ \alpha_{10} = 1,\ \alpha_{20} = \frac{1}{2},\ \alpha_{23} = \frac{1}{4},\ \beta_{00} = -\frac{1}{2}$$

and

$$f(t) = e^{-\frac{t}{2}-2}\left(-t^4 - t^3 + \frac{3}{2}t^2 - \frac{3}{2}t - \frac{3}{2}\right) + 2te^{-\frac{t}{2}-\frac{7}{4}}\left(-t^2 + \frac{5}{4}\right)$$
$$- e^{-\frac{3t}{8}-6}\left(\frac{-t^2}{16} - \frac{t}{4} + 1\right)^3 - 2\sin(t)e^{-\frac{t}{4}-2}\left(\frac{-t^2}{4} - \frac{t}{2} + 1\right).$$

The nonlinear DDE which results from the above coefficients is as follows

$$u^{(1)}(t) = -2tu\left(t - \frac{1}{2}\right) - t^2u(t) + 2\sin(t)u(\frac{t}{2}) + u^3(\frac{t}{4}) + f(t) \tag{12}$$

with the following initial condition

$$u(0) = e^{-2} \tag{13}$$

The exact solution of this problem is

$$u(t) = (-t^2 + 1 - t)e^{-\frac{t}{2}-2} \tag{14}$$

According to the Table 1, the differential transform of Eq. (12) at $t_0 = 0$ is obtained as

$$\begin{aligned}(k+1)U(k+1) = &-2\sum_{k_1=0}^{k}\sum_{s=k_1}^{N}\left[\binom{s}{k_1}\left(-\frac{1}{2}\right)^{s-k_1} U(s)\,\delta\,(k-k_1-1)\right] - \sum_{s=0}^{k}\left[U(s)\,\delta\,(k-s-2)\right] + \\ &2\sum_{s=0}^{k}\left[\frac{1}{s!}\sin\left(\frac{s\pi}{2}\right)\left(\frac{1}{2}\right)^{k-s} U(k-s)\right] + \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\left[\frac{1}{4^{k_1}}\frac{1}{4^{k_2-k_1}}\frac{1}{4^{k-k_2}}U(k_1)U(k_2-k_1)U(k-k_2)\right] + F(k),\end{aligned} \tag{15}$$

where

$$\begin{aligned}F(k) = &\ e^{-2}\sum_{s=0}^{k}\left[\frac{1}{(k-s)!}\left(-\frac{1}{2}\right)^{k-s}\left(-\delta\,(s-4) - \delta\,(s-3) + \frac{3}{2}\delta\,(s-2) - \frac{3}{2}\delta\,(s-1) - \frac{3}{2}\delta\,(s)\right)\right] \\ &+ 2e^{-\frac{7}{4}}\sum_{s=0}^{k}\left[\frac{1}{(k-s)!}\left(-\frac{1}{2}\right)^{k-s}\left(-\delta\,(s-3) + \frac{5}{4}\delta\,(s-1)\right)\right] \\ &- e^{-6}\sum_{s=0}^{k}\left[\frac{1}{(k-s)!}\left(-\frac{3}{8}\right)^{k-s}\left(-\frac{1}{4096}\delta\,(s-6) - \frac{3}{1024}\delta\,(s-5) + \frac{5}{64}\delta\,(s-3) - \frac{3}{4}\delta\,(s-1) + \delta(s)\right)\right] \\ &- 2e^{-2}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\left[\frac{1}{k_1!}\sin(\frac{k_1\pi}{2})\frac{1}{(k_2-k_1)!}\left(-\frac{1}{4}\right)^{k_2-k_1}\left(-\frac{1}{4}\delta\,(k-k_2-2) - \frac{1}{2}\delta\,(k-k_2-1) + \delta\,(k-k_2)\right)\right].\end{aligned}$$

The initial condition in Eq. (19) can be transformed as

$$U(0) = e^{-2} \tag{16}$$

By using the initial condition at Eq. (17) and solving the system of equations that can be obtained from Eq. (15) for $N = 8$ and $k = 0,1\dots,7$, we can derive the coefficients $U(k)$ and according to Eq. (9), we can obtain the 8-order approximate solution as follows

$$\begin{aligned} u(t) = e^{-2} &- 0.203002924856\,t - 0.0507507314976\,t^2 + 0.0479312470814\,t^3 - \\ &0.0137449867738\,t^4 + 0.243178632802\times10^{-2}\,t^5 - 0.314230176936\times10^{-3}\,t^6 + \\ &0.321442281989\times10^{-4}\,t^7 - 0.290305109856\times10^{-5}\,t^8. \end{aligned} \tag{17}$$

By applying the $[3,5]-$ Padé-approximation to the solution of Eq. (17), we reach to

$$\begin{aligned} u[3,5] = (&e^{-2} - 0.146011954592\,t - 0.124660846032\,t^2 \\ &+ 0.0106802836620\,t^3)/(1 + 0.421109476411\,t + 0.0855382320452\,t^2 \\ +0.0109739441614\,t^3 &+ 0.957289317417\times10^{-3}t^4 + 0.566884355876\times10^{-4}t^5) \end{aligned} \tag{18}$$

Numerical results obtained by Eqs. (17) and (18) are listed in Table 2. Plots of the exact and approximate solutions are exhibited in Figure 1.

Table 2. Numerical results of Example1

t	exact solution	GDTM solution	GDTM-Padé solution	error (GDTM)	error (GDTM-Padé)
0.0	0.135335283237	0.135335283237	0.135335283237	0	0
0.5	0.026349806141	0.026349805410	0.026349806053	7.3×10^{-10}	8.7×10^{-11}
1.0	-0.082084998624	-0.082085315482	-0.082085018441	3.2×10^{-7}	2.0×10^{-8}
1.5	-0.175801618318	-0.175812374990	-0.175801933791	1.1×10^{-5}	3.2×10^{-7}
2.0	-0.248935341840	-0.249065592895	-0.248936885149	1.3×10^{-4}	1.5×10^{-6}
2.5	-0.300500110696	-0.301399553692	-0.300503170146	9.0×10^{-4}	3.1×10^{-6}
3.0	-0.332171217645	-0.336527544757	-0.033216955293	4.4×10^{-3}	1.7×10^{-6}
3.5	-0.346886751376	-0.363404420027	-0.346856024257	1.7×10^{-2}	3.1×10^{-5}
4.0	-0.347997138885	-0.400345847857	-0.034787959618	5.2×10^{-2}	1.2×10^{-4}
4.5	-0.338775555339	-0.483433031719	-0.338464933154	1.4×10^{-1}	3.1×10^{-4}
5.0	-0.322160899608	-0.680909223823	-0.321491651365	3.6×10^{-1}	6.7×10^{-4}

Example 2: We consider Eq. (1) with $m=2, J=1,$ and the following nonzero coefficients μ_{jn}, α_{jn} and β_{jn} as

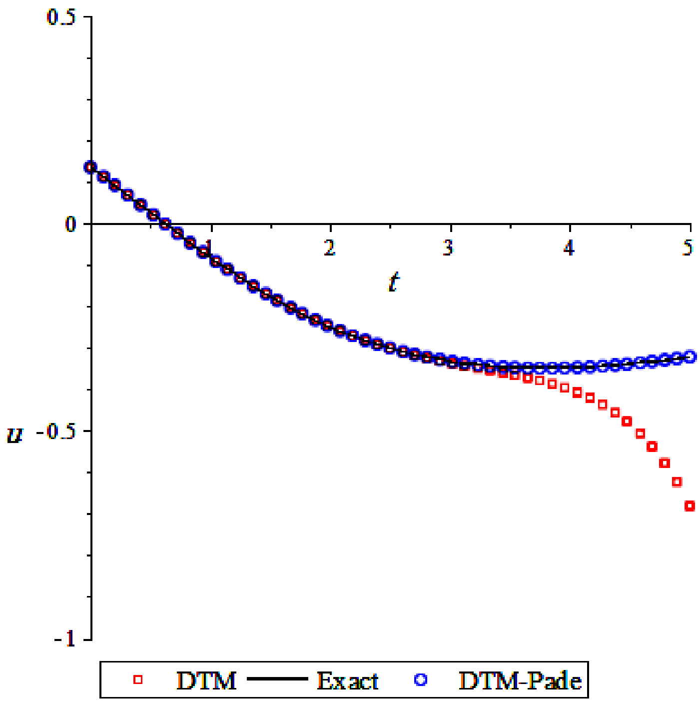

Figure1. The compared results for the GDTM solutions, GDTM-Padé solutions and exact solutions for Example 1

$$\mu_{00} = t - 1, \qquad \mu_{01} = e^{-t}, \qquad \mu_{10} = -2,$$
$$\alpha_{00} = 1, \qquad \alpha_{01} = 1, \qquad \alpha_{10} = \frac{1}{3}, \qquad \beta_{01} = -\frac{1}{5}, \qquad g(u) = u^2,$$

and

$$f(t) = -\frac{1}{4}\sin^2\left(\frac{t}{3}\right) - \frac{1}{3}\sin\left(\frac{t}{3}\right)\cos\left(\frac{t}{2}\right) - \frac{1}{9}\cos^2\left(\frac{t}{2}\right) + \left(-\frac{1}{2}\sin\left(\frac{t}{3}\right) - \frac{1}{3}\cos\left(\frac{t}{2}\right)\right)t + \frac{4}{9}\sin\left(\frac{t}{3}\right) + \frac{1}{4}\cos\left(\frac{t}{2}\right)$$
$$- e^{-t}\left(\frac{1}{6}\cos\left(\frac{t}{3} - \frac{1}{15}\right) - \frac{1}{6}\sin\left(\frac{t}{2} - \frac{1}{10}\right)\right) + \sin\left(\frac{t}{9}\right) + \frac{2}{3}\cos\left(\frac{t}{6}\right),$$

Also, the nonzero coefficients c_{in} in the initial conditions are given as

$$c_{00} = 3, \qquad c_{01} = 6, \qquad c_{10} = -2, \qquad c_{11} = 1,$$

and $\lambda_0 = 2,\ \lambda_1 = -\frac{1}{2}\cdot$

These coefficients result in the following nonlinear delay differential problem

$$u^{(2)}(t) = e^{-t}u^{(1)}\left(t - \frac{1}{5}\right) + (t-1)u(t) - 2u\left(\frac{t}{3}\right) + u^2(t) + f(t) \tag{19}$$

with the initial conditions

$$\begin{cases} 3\,u(0) + 6\,u^{(1)}(0) = 2\,, \\ -2\,u(0) + u^{(1)}(0) = -\dfrac{1}{2} \end{cases} \tag{20}$$

The exact solution of this problem is

$$u(t) = \frac{1}{2}\sin\left(\frac{t}{3}\right) + \frac{1}{3}\cos\left(\frac{t}{2}\right) \tag{21}$$

The differential transform of Eq. (19) at $t_0 = 0$ is as follows

$$(k+1)(k+2)U(k+2) = \sum_{k_1=0}^{k}\sum_{s=k_1}^{N-1}\left[\frac{(-1)^{k-k_1}}{(k-k_1)!}\binom{s}{k_1}\left(-\frac{1}{5}\right)^{s-k_1}(s+1)U(s+1)\right] + \sum_{s=0}^{k}\left[U(k-s)\,\delta(s-1)\right] \tag{22}$$
$$-U(k) - \frac{2}{3^k}U(k) + \sum_{s=0}^{k}\left[U(s)U(k-s)\right] + F(k),$$

where

$$\begin{aligned} F(k) = &-\frac{1}{4}\sum_{s=0}^{k}\left[\frac{1}{s!}\left(\frac{1}{3}\right)^{s}\sin\left(\frac{s\pi}{2}\right)\frac{1}{(k-s)!}\left(\frac{1}{3}\right)^{k-s}\sin\left(\frac{(k-s)\pi}{2}\right)\right] \\ &-\frac{1}{3}\sum_{s=0}^{k}\left[\frac{1}{s!}\left(\frac{1}{3}\right)^{s}\sin\left(\frac{s\pi}{2}\right)\frac{1}{(k-s)!}\left(\frac{1}{2}\right)^{k-s}\cos\left(\frac{(k-s)\pi}{2}\right)\right] \\ &-\frac{1}{9}\sum_{s=0}^{k}\left[\frac{1}{s!}\left(\frac{1}{2}\right)^{s}\cos\left(\frac{s\pi}{2}\right)\frac{1}{(k-s)!}\left(\frac{1}{2}\right)^{k-s}\cos\left(\frac{(k-s)\pi}{2}\right)\right] \\ &+\sum_{s=0}^{k}\left[\delta(k-s-1)\left(-\frac{1}{2}\frac{1}{s!}\left(\frac{1}{3}\right)^{s}\sin\left(\frac{s\pi}{2}\right)-\frac{1}{3}\frac{1}{s!}\left(\frac{1}{2}\right)^{s}\cos\left(\frac{s\pi}{2}\right)\right)\right]+\frac{4}{9}\frac{1}{k!}\left(\frac{1}{3}\right)^{k}\sin\left(\frac{k\pi}{2}\right) \\ &+\frac{1}{4}\frac{1}{k!}\left(\frac{1}{2}\right)^{k}\cos\left(\frac{k\pi}{2}\right)-\sum_{s=0}^{k}\left[\frac{(-1)^{k-s}}{(k-s)!}\left(\frac{1}{6}\frac{1}{s!}\left(\frac{1}{3}\right)^{s}\cos\left(\frac{s\pi}{2}-\frac{1}{15}\right)-\frac{1}{6}\frac{1}{s!}\left(\frac{1}{2}\right)^{s}\sin\left(\frac{s\pi}{2}-\frac{1}{10}\right)\right)\right] \\ &+\frac{1}{k!}\left(\frac{1}{9}\right)^{k}\sin\left(\frac{k\pi}{2}\right)+\frac{2}{3}\frac{1}{k!}\left(\frac{1}{6}\right)^{k}\cos\left(\frac{k\pi}{2}\right) \end{aligned}$$

Also, Eq. (20) is transformed as

$$U(0) = u(0) = \frac{1}{3},\quad U(1) = u^{(1)}(0) = \frac{1}{6} \tag{23}$$

Using the Eqs. (22) and (23) and by taking $N = 5\,, k = 0,1,2,3$, the coefficients $U(k)$ are determined. Then, by using the inverse transformation rule (Eq. (9)), the 5-order approximate solution is obtained as follows

$$u(t) = \frac{1}{3} + \frac{1}{6}t - 0.0416666665130\,t^2 - 0.00308640088768\,t^3 + 0.867875952857\times10^{-3}t^4 \tag{24}$$
$$+0.179246289966\times10^{-4}t^5$$

By applying the $[3,3]$ Padé-approximation to the solution of Eq. (24), we reach to

$$u[3,3] = \frac{\frac{1}{3} + 0.320139569072\,t + 0.0419625583936\,t^2 - 0.0159946118382\,t^3}{1 + 0.460418707215\,t + 0.0206783211124\,t^2 + 0.00848854478165\,t^3} \tag{25}$$

Numerical results obtained by these approximations are shown in Table 3. For better comparison, the exact and approximate solutions for this example are plotted in Figure 2.

Table 3. Numerical results of Example 2

t	exact solution	GDTM solution	GDTM-Padé solution	error (GDTM)	error (GDTM-Padé)
0.0	0.333333333333	0.333333333333	0.333333333333	0	0
0.5	0.405918873584	0.405919002319	0.405918952945	1.3×10^{-7}	7.9×10^{-8}
1.0	0.456124869028	0.456132733181	0.456127436534	7.9×10^{-6}	2.6×10^{-6}
1.5	0.483609058927	0.483696467845	0.483619389813	8.7×10^{-5}	1.0×10^{-5}
2.0	0.489285670158	0.489768396887	0.489271466337	4.8×10^{-4}	1.4×10^{-5}
2.5	0.475195880730	0.477010176884	0.474956706841	1.8×10^{-3}	2.4×10^{-4}
3.0	0.444314559627	0.449654147777	0.443246065237	5.3×10^{-3}	1.1×10^{-3}
3.5	0.400307137745	0.413570550237	0.397090388751	1.3×10^{-2}	3.2×10^{-3}
4.0	0.347253385000	0.376334743008	0.339533651100	2.9×10^{-2}	7.7×10^{-3}
4.5	0.289356285728	0.347294420276	0.273488796880	5.8×10^{-2}	1.6×10^{-2}
5.0	0.230656107027	0.337636829046	0.201586995190	1.1×10^{-1}	2.9×10^{-2}

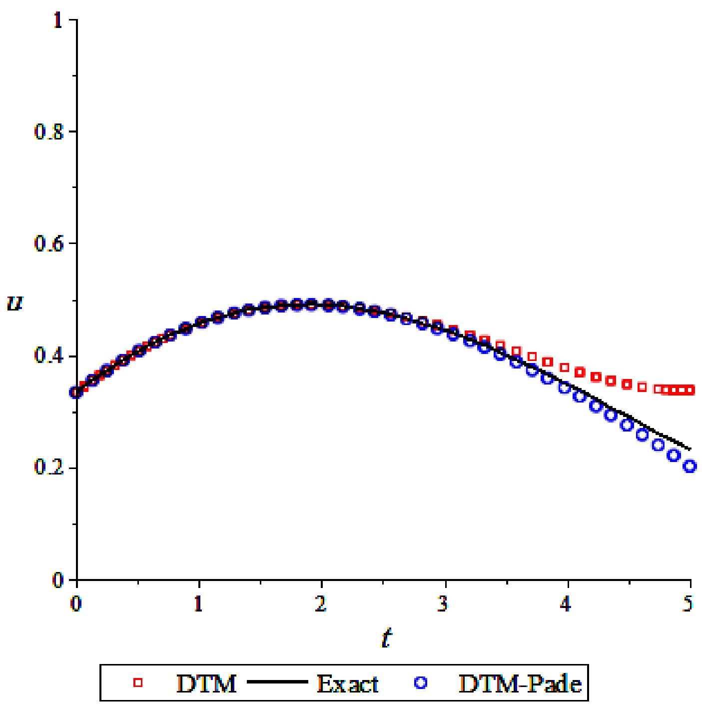

Figure 2. The compared results for the GDTM solution, GDTM-Padé solutionand exact solution for Example 2

Note that for both aforementioned examples, the GDTM solutions in the interval $[0,1]$ are more accurate than the approximation solutions obtained in Ref. [21]. When $t > 1$, the accuracy of the approximation is improved significantly by using Padé-approximation for GDTM solution.

Example 3: In this example, consider Eq. (1) with $m = 3, J = 1,$ and the following nonzero coefficients μ_{jn}, α_{jn} and β_{jn}

$$\mu_{01} = e^{t-1}, \qquad \mu_{11} = -1, \qquad \mu_{02} = \frac{t}{3},$$
$$\alpha_{01} = \frac{1}{3}, \qquad \alpha_{11} = \frac{1}{2}, \qquad \alpha_{02} = 1,$$

and

$$f(t) = -e^{t-1}\left(-\frac{8}{2187}t^7 + \frac{2}{27}t^5 + \frac{5}{81}t^4 - \frac{16}{27}t^3 + \frac{1}{3}t^2 - \frac{4}{3}t + 3\right) + \frac{893}{48}t^7 - \frac{5847}{16}t^5 - \frac{305}{48}t^4 + 374t^3 + \frac{235}{4}t^2 - \frac{290}{3}t + 9,$$

The nonzero coefficients c_{in} in the initial conditions are given as

$$c_{00} = 1, \qquad c_{01} = -1, \qquad c_{02} = -2, \qquad c_{11} = 1, \qquad c_{12} = -1, \qquad c_{21} = 2, \qquad c_{22} = 3,$$

and

$$\lambda_0 = 5, \qquad \lambda_1 = 7, \qquad \lambda_2 = -6.$$

Therefore, we arrive at the following linear DDE as

$$u^{(3)}(t) = \frac{t}{3}u^{(2)}(t) + e^{t-1}u^{(1)}\left(\frac{t}{3}\right) - u^{(1)}\left(\frac{t}{2}\right) + f(t) \tag{26}$$

with the initial condition

$$\begin{cases} u(0) - u^{(1)}(0) - 2\,u^{(2)}(0) = 5 \\ u^{(1)}(0) - u^{(2)}(0) = 7 \\ 2\,u^{(1)}(0) + 3\,u^{(2)}(0) = -6 \end{cases} \tag{27}$$

The exact solution of this problem is

$$u(t) = -t^8 + 3t^6 + t^5 - 4t^4 + t^3 - 2t^2 + 3t \tag{28}$$

The differential transform of Eq. (26) at $t_0 = 0$ is obtained as

$$(k+1)(k+2)(k+3)U(k+3) = \frac{1}{3}\sum_{s=0}^{k}\Big[(s+1)(s+2)U(s+2)\,\delta(k-s-1)\Big] + e^{-1}\sum_{s=0}^{k}\left[\frac{1}{(k-s)!}\frac{1}{3^s}(s+1)U(s+1)\right] - \frac{1}{2^k}(k+1)U(k+1) + F(k) \tag{29}$$

where

$$F(k) = -e^{-1}\sum_{s=0}^{k}\left[\frac{1}{(k-s)!}\left(\begin{array}{l} -\frac{8}{2187}\delta(s-7) + \frac{2}{27}\delta(s-5) + \frac{5}{81}\delta(s-4) \\ -\frac{16}{27}\delta(s-3) + \frac{1}{3}\delta(s-2) - \frac{4}{3}\delta(s-1) + 3\,\delta(s) \end{array}\right)\right] + \frac{893}{48}\delta(k-7) - \frac{5847}{16}\delta(k-5) - \frac{305}{48}\delta(k-4) + 374\,\delta(k-3) + \frac{235}{4}\delta(k-2) - \frac{290}{3}\delta(k-1) + 9\,\delta(k)$$

The initial condition in Eq. (27) can be transformed as

$$U(0) = u(0) = 0,\;\; U(1) = u^{(1)}(0) = 3,\;\; U(2) = \frac{u^{(2)}(0)}{2} = -2 \tag{30}$$

By taking $N = 8$ and $k = 0,1\ldots,5$, we can derive the coefficients $U(k)$ and get the 8-order approximation solution as

$$u(t) = 3t - 2t^2 + t^3 + 4t^4 + t^5 + 3t^6 - 2.68005974139\times 10^{-15}t^7 - 0.999999999999\,t^8$$

It is observed that the GDTM solution for this example is very close to the exact solution. For this example, the numerical results are shown in Table 4 and the exact and approximate solutions are plotted in Figure 3.

Table 4. Numerical results of example3

t	exact solution	GDTM solution
0.0	0.00000000000	0.00000000000
0.5	0.94921875000	0.94921875000
1.0	1.00000000000	1.00000000000
1.5	−0.73828125000	−0.73828125000
2.0	−90.0000000000	−90.0000000000
2.5	−841.425781250	−841.425781250
3.0	−4437.00000000	−4436.99999999
3.5	−17050.1132812	−17050.1132812
4.0	−53204.0000000	−53203.9999999
4.5	$-1.42970800781\times 10^5$	$-1.42970800781\times 10^5$
5.0	$-3.43035000000\times 10^5$	$-3.43035000000\times 10^5$

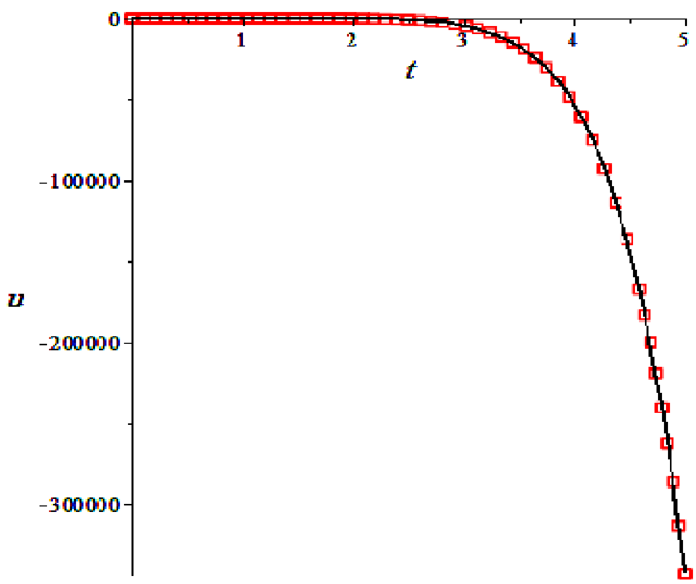

Figure 3. The compared results for GDTM and exact solutions for Example 3

5. Conclusions

In this study, the GDTM-Padé technique is employed successfully for solving DDEs. GDTM reduces the computational difficulties and needs fewer computations in comparison with traditional methods. Also, Padé-approximation is applied to improve the accuracy of GDTM solutions. Three examples solved and the results have been compared with the exact solution through plotting figures and giving tables. These comparisons verify the accuracy of the proposed method. Comparing with the results reported by the homotopy perturbation method [21], the solution obtained by GDTM-Padé solutions is in a good agreement with exact solution meets higher accuracy. Thus, the GDTM-Padé method is an effective method for solving linear and nonlinear DDEs.

References

[1] Baker C. T. H., Paul C. A. H., Wille D. R.: Issues in the Numerical Solution of Evolutionary Delay Differential Equations. Numerical Analysis Report No. 248. University of Manchester. (1994).

[2] Driver R. D.: Ordinary and Delay Differential Equations, New York: Springer-Verlag, New York, Inc. (1977).

[3] Kuang Y.: Delay Differential Equations with Applications in Population Dynamics, Academic Press, (1993).

[4] Macdonald N.: Biological Delay System: Linear Stability Theory, Cambridge University Press, (1989).

[5] Zhou J. K.: Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China, (1986).

[6] Chen C. K., Ho S. H.: Solving partial differential equations by two dimensional differential transform method. Applied Mathacis and Computation, **106,** 171-179 (1999).

[7] Ayaz F.: Solutions of the systems of differential equations by differential transform method. Applied Mathematics and Computation, **147**, 547-567 (2004).

[8] Chen C. K.: Solving partial differential equations by two dimensional differential transformation method. Applied Mathematics and Computation, **106,** 171-179 (1999).

[9] Jang M. J., Chen C. K.: Two-dimensional differential transformation method for partial differential equations. Applied Mathemacis and Computation, **121**, 261-270 (2001).

[10] Kangalgil F., Ayaz F.: Solitary wave solutions for the KDV and mKDV equations by differential transformation method. Choas Solitons and Fractals, **41**, 464-472(2009).

[11] Arikoglu A., Ozkol I.: Solution of fractional differential equations by using differential transformation method. Chaos Solitons and Fractals, **34**, 1473-1481 (2007).

[12] Arikoglu A., Ozkol I.: Solution of difference equations by using differential transformation method. Applied Mathemacis and Computation, **174**, 1216-1228(2006).

[13] Odibat Z. M.: Differential transformation method for solving Volterra integral equations with seperable kernels. Mathematical and Computer Modelling, **48**, 1144-1149 (2008).

[14] Ravi Kanth A. S. V., Aruna K.: Differential transform method for solving linear and non-linear systems of patrial differential equations. Physics Letters A, **372**, 6896-6898 (2008).

[15] Ravi Kanth A. S. V., Aruna K.: Differential transform method for solving the linear and nonlinear Klein-Gordon equation. Computer Physics Communications, **180**, 708-711(2009).

[16] Jang M. J., Chen C. L., Liu Y. C.: Two-dimensional differential transform for partial differential equations. Applied Mathemacis and Computation,**121**, 261-270(2001).

[17] Abdel-Halim Hassan I. H.: Different applications for the differential transformation in the differential equations. Applied Mathematics and Commputation, **129**, 183-201(2002).

[18] Boyd J.: Padé approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain.Computers in Physics,**11**, 299-303 (1997).

[19] Baker G. A., Graves–Morris P.: Padé Approximants, Cambridge U.P., (1996).

[20] Baker G. A.: Essential of Padé Approximants, Academic Press, London, UK, (1975).

[21] Shakeri F., Dehghan M.: Solution of delay differential equations via a homotopy perturbation method. Mathematical and Computer Modelling, **48**, 486–498 (2008).

[22] Karakoc F., Bereketoglu H.: Solutions of delay differential equations by using differential transform method, International Journal of Computer Mathematics, **86**, 914–923(2009).

[23] Smarda Z., Diblik J., Khan Y.: Extension of the differential transformation method to nonlinear differential and integro-differential equations with proportional delays, Advances in Difference Equations, **69**, 1-12(2013).